From localization formulas to orbital integrals

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TRAVELLING WITH MICHÈLE

FROM REPRESENTATIONS AND HARMONIC ANALYSIS ON LIE GROUPS TO INDEX THEORY

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- $X_K = (K = 0)$ smooth submanifold.

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$e_K\left(N_{X_K/X}\right)$ equivariant Euler class of $N_{X_K/X}$.

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- On $X \setminus X_K, 1 = d^K \frac{\alpha}{d_K\alpha}$.
- Use Stokes formula.

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- $$\underbrace{\int_X \mu|_{t=+\infty}}_{\text{global}} \xrightarrow{\int_X \alpha_t \mu|_{t>0}} \underbrace{\int_{X_K} \frac{\mu}{e_K\left(N_{X_K/X}\right)}}_{\text{local}}|_{t=0}.$$

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- Atiyah-Bott fixed point formula, $\chi(g) = \int_{X_g} \widehat{A}_g(TX)\,\mathrm{ch}_g(E)$.

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- Atiyah-Witten: index theorem formal consequence of localization on loop space LX w.r.t. Killing vector field $Z(x) = \dot{x}$ (d_K supersymmetry).
- I had shown how the heat equation method provides the proper proof of localization in this infinite dimensional context.

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- Example: generic coadjoint orbits G/T.

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- My formal answer: K^X lifts to K^{LX} on LX, and $\left[K^{LX}, Z\right] = 0 \ldots$
- …so that one should (formally!) use localization with two commuting Killing vector fields instead of one.

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- Make $t \to 0$ and get Kirillov like formula.
- Nicole Berline and Michèle Vergne's reaction: 'Now, you should prove the families index theorem!'

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- ...and gives families index theorem in this special case.

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- Compare with $\sqrt{t} D^X + c\left(K^X\right)/4\sqrt{t}$.

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2. As $t \to 0$,

$$\operatorname{ch}(A_t) \to \pi_*\left[\widehat{A}\left(TX, \nabla^{TX}\right)\operatorname{ch}\left(E, \nabla^E\right)\right]$$

local version of AS families index theorem.

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- Right-hand side orbital integrals for $\mathrm{SL}_2\left(\mathbf{R}\right)$.

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Example

$G = \mathrm{SL}_2(\mathbf{R})$, $K = S^1$, X upper half-plane, $TX \oplus N$ of dimension 3.

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- If $Z = \Gamma \setminus X$, orbital integrals part of trace of heat kernel.

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- $X(\gamma) \subset X$ totally geodesic.

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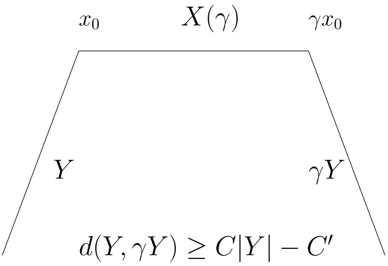

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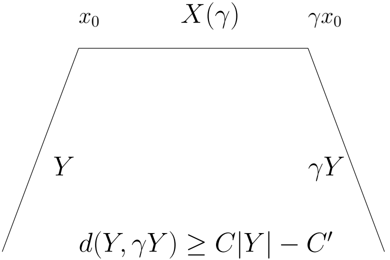

$$p_t^X(x, x') \leq C \exp(-C' d^2(x, x')).$$

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$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^{\mathfrak{g},X} - c\right)/2\right)\right] = \frac{\exp\left(-\left|a\right|^2/2t\right)}{(2\pi t)^{p/2}}$$

$$\int_{i\mathfrak{k}(\gamma)} \mathcal{J}_{\gamma}\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}^{E}\left[k^{-1} e^{-Y_0^{\mathfrak{k}}}\right] \exp\left(-\left|Y_0^{\mathfrak{k}}\right|^2/2t\right) \frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

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Theorem (B. 2011)

If $\gamma = e^a k^{-1}, a \in \mathfrak{p}, k \in K, \mathrm{Ad}(k) a = a$, there is an explicit function $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right), Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$, such that

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^{\mathfrak{g},X} - c\right)/2\right)\right] = \frac{\exp\left(-|a|^2/2t\right)}{(2\pi t)^{p/2}}$$

$$\int_{i\mathfrak{k}(\gamma)} \mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}^E\left[k^{-1} e^{-Y_0^{\mathfrak{k}}}\right] \exp\left(-\left|Y_0^{\mathfrak{k}}\right|^2/2t\right) \frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

Localization from G to $\mathfrak{g}$.

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Definition

$$\mathcal{J}_\gamma\left(Y_0^\mathfrak{k}\right) = \frac{1}{\left|\det\left(1-\operatorname{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\left(\operatorname{ad}\left(Y_0^\mathfrak{k}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(\operatorname{ad}\left(Y_0^\mathfrak{k}\right)_{\mathfrak{k}(\gamma)}\right)}$$

$$\left[\frac{1}{\det\left(1-\operatorname{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_0^\perp(\gamma)}}\right.$$

$$\left.\frac{\det\left(1-\operatorname{Ad}\left(k^{-1}e^{-Y_0^\mathfrak{k}}\right)\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-\operatorname{Ad}\left(k^{-1}e^{-Y_0^\mathfrak{k}}\right)\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

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$$\mathrm{Tr_s}\left[P_t^X\left(x,x\right)\right] = e^{-ct/2}\frac{1}{\left(2\pi t\right)^{p/2}} \int_{i\mathfrak{k}} J_1\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^E\left[e^{-Y_0^{\mathfrak{k}}}\right]\exp\left(-\left|Y_0^{\mathfrak{k}}\right|^2/2t\right)\frac{dY_0^{\mathfrak{k}}}{\left(2\pi t\right)^{q/2}}.$$

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- $$\underbrace{L\left(g\right)|_{t=+\infty}}_{\text{global}} \xrightarrow{\mathrm{Tr_s}\left[g\exp\left(-tD^{X,2}\right)\right]|_{t>0}} \underbrace{\int_{X_g}\widehat{A}_g\left(TX\right)\mathrm{ch}_g\left(E\right)}_{\text{local}}|_{t=0}.$$

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- $$\underbrace{\operatorname{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X}\right)\right]_{b=0}}_{\text{global orbital integral}} \xrightarrow{\operatorname{Tr_s}^{[\gamma]}\left[g\exp\left(-tD_b^{R,2}\right)\right]|_{b>0}} \underbrace{\text{Geom. formula}|_{b=+\infty}}_{\text{local via Lie algebra}}$$

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- ...which is the total space $\widehat{\mathcal{X}}$ of $TX \oplus N$ over $X = G/K$.
- Two separate constructions on G and on $\mathfrak{g}$.

Dirac, Casimir and Kostant

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Remark

$\widehat{D}^{\mathrm{Ko}}$ acts on $C^{\infty}\left(G, \Lambda^{\cdot}\left(\mathfrak{g}^{*}\right)\right)$, while $C^{\mathfrak{g}}$ acts on $C^{\infty}\left(G, \mathbf{R}\right)$.

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Remark

Using the fiberwise Bargmann isomorphism, $\mathcal{L}_b^X$ acts on

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- $b \to +\infty$, geodesic f. $\nabla_{Y^{TX}}$ dominates $\Rightarrow$ closed geodesics.

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Theorem (B. 2011)

For $b > 0, t > 0$,

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- After rescaling of Y^{TX}, Y^N, as $b \to +\infty$,
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- The Duflo-Harish Chandra isomorphism appears explicitly in our formula.

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Toutes mes amitiés, Michèle!